Mean-field analysis of phase transitions in the emergence of hierarchical society

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Emergence of hierarchical society is analyzed by use of a simple agent-based model. We extend the mean-field model of Bonabeau *et al.* [Physica A **217**, 373 (1995)] to societies obeying complex diffusion rules where each individual selects a moving direction following their *power rankings*. We apply this mean-field analysis to the pacifist society model recently investigated by use of Monte Carlo simulation [Physica A **367**, 435 (2006)]. We show analytically that the self-organization of hierarchies occurs in two steps as the individual density is increased and there are three phases: one egalitarian and two hierarchical states. We also highlight that the transition from the egalitarian phase to the first hierarchical phase is a continuous change in the order parameter and the second transition causes a discontinuous jump in the order parameter.

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I. INTRODUCTION

The emergence of hierarchies is found in a wide range of societies or animal clusters. In these phenomena, a small difference in each individual is enhanced by some causes, and classes are organized spontaneously.

In the seminal work of Bonabeau *et al.*, they have shown that a hierarchical society can emerge in a simple agent-based model [1]. In their model each individual is assumed to have *power* and diffuses on the square lattice. If two individuals meet, they fight and then the winner increases its power, while the power of the loser decreases. The winning probability for the stronger individual is set larger than the weaker individual, therefore repeated battle causes enhancement of the power difference. In competition with this battle effect, the powers of all individuals relax toward zero gradually. This model was found to exhibit a transition from the homogeneous equal society to a heterogeneous hierarchical society [1].

There are several modifications of the Bonabeau model. Stauffer *et al.* investigated some models where feedback effect from a structure of hierarchy was included into the winning probability and a multiplicative relaxation was employed [2–6]. Generalization of the space where individuals move has also been investigated. Naumis *et al.* introduced value into sites so that each site is attractive or not, and found that the transition strongly depends on a distribution of attractive sites [7,8]. Investigations of the Bonabeau model on scale-free networks [9] or a fully connected graph [10] have also been developed in recent years.

Recently, Odagaki *et al.* have proposed variations of the Bonabeau model, where diffusion algorithms are modified to include an effect of a trend of society [11,12]. They have shown by Monte Carlo simulations that complex hierarchical transitions occur in these models. Especially, in the timid society where individuals are pacifist, the self-organization of hierarchies occurs in two steps and there are three phases: one egalitarian phase and two hierarchical phases. Interestingly, while the transition from the egalitarian phase to the first hierarchical phase is continuous, the second transition shows a discontinuous jump in the order parameter [11].

Inspired by the simulation results, analytical approaches have been developed. Lacasa *et al.* introduced the mean-field approximation in several models based on the Bonabeau model, and explained the phase transition by bifurcation theory [13]. Ben-Naim *et al.* investigated a time evolution of power distribution for a model, which is slightly different from the Bonabeau model, and the phase transition was found analytically [14,15]. However, these works were not concerned about generalization of diffusion rules and therefore one cannot apply them to complex transitions which appear in works of Odagaki *et al.* [11,12].

In this paper, we introduce a generalized mean-field analysis. This mean-field analysis can deal with the emergence of hierarchies where the diffusion rule is modified from the original Bonabeau model. An important idea to understand complex phase transition is *power ranking*. By use of an agent interaction which depends on the power ranking, we investigate the phase transition analytically. We also demonstrate that this mean-field analysis can explain the successive phase transitions investigated in the simulation of the timid society [11].

We organize this paper as follows. We explain the agent model we employ, in Sec. II. In Sec. III we investigate this model by mean-field analysis where the agent interactions are described as a function of power ranking. We also show applications of our analysis for several model societies in Sec. IV. Section V is devoted to discussion.

II. MODEL

A target of our mean-field analysis is a group of stochastic models which show self-organization similar to the emergence of hierarchies. These models are based on the pioneering work of Bonabeau *et al.* [1]. In these models, fighting between individuals diffusing on a square lattice causes the emergence of hierarchies. Each individual has power and it increases or decreases by a win or a loss in the fighting. The essential processes of these models are diffusion, fighting, and relaxation of the power.

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We consider N individuals diffusing on an $L\times L$ square lattice with periodic boundary condition, where every lattice site is occupied at most by one individual. Every individual has power F as an internal degree of freedom. At each Monte Carlo step individuals move on the sites and interact with each other according to the following protocol. At first, randomly selected individual i moves to one of the nearest neighbor sites. When i tries to move to a site already occupied by individual j, i and j fight each other. If i wins, i and j exchange their positions, and if i loses, they keep their original positions. As a result of fighting, the power increases by unity for every victory and decreases by unity for every loss. The probability $P_{i,j}$ that i wins the fight against j is determined by the difference of their powers F_i and F_j. After every individual is accessed once for movement, powers of all individuals relax toward zero.

Various societies can be modeled by specifying three rules in the protocol: diffusion, fighting, and relaxation. In the original Bonabeau model, these rules were as follows. First, each individual moves to one of the nearest neighbor sites with equal probability, which is equal to 1/4 in the square lattice. Second, the probability $P_{i,j}$ is assumed as

$$P_{i,j} = \frac{1}{1+\exp[-\eta(F_i - F_j)]}, \tag{1}$$

where η is introduced as a controlling parameter. When $\eta \to \infty$, the stronger one always wins a fight and when $\eta=0$, the winning probabilities of both ones are equal. Finally, at the relaxation stage, $F_i(t+1)$ at time $t+1$ is given by $F_i(t)$ through

$$F_i(t+1) = F_i(t) - \mu \tanh[F_i(t)]. \tag{2}$$

Here the unit of time is defined by one Monte Carlo step and μ represents an additional controlling parameter. This relaxation rule indicates that people lose their powers by a constant amount when their power is sufficiently large. Bonabeau *et al.* have shown that increasing a number density $\rho = N/L^2$, the phase transition from an egalitarian society, where F_i is equal for all i, to a hierarchical society happens at a critical density.

In addition to the *power*, we characterize each individual by the winning fraction X_i of fighting history, which is defined by

$$X_i = \frac{D_i}{D_i + S_i}, \tag{3}$$

where D_i and S_i are the number of fights won and lost, respectively. In order to characterize the static status of a society, we employ the variance of X_i as the order parameter as follows:

$$\sigma^2 = \langle (X_i - \langle X_i \rangle)^2 \rangle, \tag{4}$$

where $\langle \cdots \rangle$ represents the average over all individuals, $\frac{1}{N}\Sigma_i \cdots$. Note that $\sigma=0$ corresponds to an egalitarian state and $\sigma \neq 0$ in a hierarchical state. Stauffer *et al.* have introduced a different order parameter that is a variance of winning probability $P_{i,j}$ as follows:

$$\tilde{\sigma}^2 = \langle (P_{i,j} - \langle P_{i,j} \rangle)^2 \rangle. \tag{5}$$

In this case the average $\langle \cdots \rangle$ is performed over all fights. σ and $\tilde{\sigma}$ are slightly different but both of them can catch an emergence of hierarchy. Because we are also interested in a profile of winning fraction, we use σ in this paper.

Several authors have investigated some models where Eqs. (1) and (2) are modified [2–5]. However, here we consider models where only diffusion rule is modified and the other rules are the same as Eqs. (1) and (2) such as works of Odagaki *et al.* [11,12]. In their model, individuals decide their moving directions from a relative power relationship of nearest neighbors.

In the subsequent two sections, we will investigate the phase transitions analytically by use of a mean-field approach where agent interactions depend on the power ranking. One can calculate the average gain from battles as a function of the ranking. We will show that the qualitative nature of power profile changes in two steps from the competition between the average gain and the power relaxation. One can understand the origin of successive transitions based on the gain function.

III. MEAN-FIELD ANALYSIS

A. Mean-field equation

In order to analyze the phase transition we introduce a mean-field analysis where an individual interacts with the other and evolution of physical quantities is described in the continuous time limit. Here we take this limit so that the number of fights and the amount of power relaxation per unit time are constant.

In this mean-field analysis, the time evolution of power F_i is expressed as

$$\frac{dF_i}{dt} = \frac{dD_i}{dt} - \frac{dS_i}{dt} - \mu \tanh(F_i). \tag{6}$$

Here D_i and S_i represent the number of wins and losses since $t=0$, respectively. The last term of the right-hand side expresses an effect of the relaxation. Time evolutions of D_i and S_i depend on the characteristics of the model society, and then they can be written as

$$\frac{dD_i}{dt} = \sum_{j\neq i}^{N} P_{i,j} Q_{i,j}, \quad \frac{dS_i}{dt} = \sum_{j\neq i}^{N} (1 - P_{i,j}) Q_{i,j}. \tag{7}$$

Here, $P_{i,j}$ denotes the probability that i wins in a fight against j, and it is supposed that $P_{i,j}$ is a function of power difference $\Delta F_{i,j} = F_i - F_j$. The probability $P(\Delta F_{i,j})$ must satisfy three natural conditions as follows:

$$P(0) = 1/2,$$

$$P(\Delta F_{i,j}) \to \begin{cases} 1 & (\Delta F_{i,j} \to \infty) \\ 0 & (\Delta F_{i,j} \to -\infty), \end{cases}$$

$$P(\Delta F_{i,j}) = 1 - P(\Delta F_{j,i}). \tag{8}$$

In Eq. (7), $Q_{i,j}$ is the frequency of a battle between i and j and it reflects a trend of the model society.

This approach is a generalization of the mean-field model of Bonabeau *et al.* [1]. In their model $Q_{i,j}$ is equal to $2\rho/(N-1)$ independent of its suffix. To deal with more general cases, we investigate the case where $Q_{i,j}$ is a function of *power rankings* of individuals i and j, which is appropriate to the timid or the challenging society of Odagaki *et al.* [11,12]. Here, power ranking of individual i contains the information about the number of stronger individuals than i and that of weaker individuals. Concrete examples of $Q_{i,j}$ will be discussed in the next section.

In this approach, we impose two characteristics to $\{Q_{i,j}\}$. According to the model introduced by [11], we assume that $Q_{i,j}$ only depends on power rankings of individuals i and j, as well as the number density of individuals, ρ. Here, the power ranking of individual i has three sorts of information. The first is the number of individuals weaker than i, N_i^w. The second is the number of individuals who have equivalent power as i, N_i^e (excluding oneself), and the third is the number of individuals stronger than i, N_i^s. Note that only two of these three numbers are independent because the sum of these is constant: $N_i^w+N_i^e+N_i^s=N-1$. Here we express a set of these three numbers as $R_i=\{N_i^w,N_i^e,N_i^s\}$, and explicitly write this dependence of $Q_{i,j}$ as $Q_{i,j}=Q(R_i,R_j;\rho)$.

Second, we need to determine a condition of $Q(R_i,R_j;\rho)$ when $N_i^e\neq 0$. We define such $Q(R_i,R_j;\rho)$ as the limit of infinitesimal power difference. Suppose that there are N_i^e individuals whose rankings are the same as individual i. Let us define $\boldsymbol{G}$ as a group of these N_i^e+1 individuals. The total frequency of battles between this group $\boldsymbol{G}$ and another individual j is equal to $(N_i^e+1)Q(R_i,R_j;\rho)$. On the other hand, suppose that powers of these (N_i^e+1) individuals slightly fluctuate and their rankings become no longer equivalent. We express rankings for such a situation as R_k' for $k=1,2,\ldots,N$. If power fluctuations are small enough, power relationships between group $\boldsymbol{G}$ and the other individuals are the same as before so that $R_k'=R_k$ for $k\notin\boldsymbol{G}$. We request that the total battle frequency between group $\boldsymbol{G}$ and individual $j(\notin\boldsymbol{G})$ is conserved after such fluctuations as follows:

$$Q(R_i,R_j;\rho)=\frac{1}{N_i^e+1}\sum_{k\in\boldsymbol{G}}Q(R_k',R_j;\rho). \tag{9}$$

For $k\in\boldsymbol{G}$, $R_k'=\{N_k'^w,N_k'^e=0,N_k'^s=N-1-N_k'^w\}$ and $N_k'^w$ ranges from N_i^w to $(N_i^w+N_i^e)$. In the same sense, we also impose the conservation of battle frequency within the group $\boldsymbol{G}$ as follows:

$$Q(R_i,R_j;\rho)=\frac{2}{(N_i^e+1)N_i^e}\sum_{l\in\boldsymbol{G}}\sum_{k\in\boldsymbol{G},k>l}Q(R_l',R_k';\rho) \tag{10}$$

for $i,j\in\boldsymbol{G}$ so that $R_j=R_i$.

The emergence of a hierarchical society is measured from the distribution of winning fractions defined in Eq. (3). We employ the variance σ^2 of the winning fraction distribution as the order parameter, and cast a spotlight on its dependence on the individual number density ρ.

B. Steady state

In order to discuss the phase transition, we focus on the steady states in the long time limit where time derivatives of winning fractions are equal to zero: $dX_i(t)/dt=0$. Note that this condition does not necessarily mean an invariance of power distribution. As shown in this subsection, although winning fractions are invariant in time, powers of some individuals actually increase or decrease along the time evolution in a hierarchical society state.

To see conditions for powers in a steady state, we analyze Eq. (6) in detail. For simplicity, we renumber individuals in the steady state as

$$F_1\leq F_2\leq\cdots\leq F_{N-1}\leq F_N, \tag{11}$$

without loss of generality. Combining with the identity, $D_i-S_i=(D_i+S_i)(2X_i-1)$, we can reexpress the first term of the right-hand side of Eq. (6) for the steady state as follows:

$$\frac{dD_i}{dt}-\frac{dS_i}{dt}=(2X_i-1)\frac{d}{dt}(D_i+S_i). \tag{12}$$

From Eq. (7), $d(D_i+S_i)/dt$ depends only on $\{Q_{i,j}\}$ and does not depend on $\{P_{i,j}\}$. Since the power ranking must be constant in the steady states, $Q_{i,j}$ is also invariant in time. Therefore we can get a closed equation for F_i from Eqs. (7) and (12) as follows:

$$\frac{dF_i}{dt}=H_i-\mu\tanh(F_i). \tag{13}$$

Here, $H_i\equiv dD_i/dt-dS_i/dt$ represents a gain from battles and does not depend on time.

Equation (13) has two kinds of asymptotic solutions: in one solution F_i is constant, and in the other F_i is linearly proportional to t. The former solution satisfies $\tanh F_i=H_i/\mu$. This equation is not easily solved because H_i can depend on F_j of any j. However, there is always one trivial solution in which all individuals have the same power $F_i=0$. In this situation, X_i is a half for all individuals and therefore $H_i=0$ from Eq. (12). The latter solution is easily checked by ansatz as follows:

$$F_i=k_it+c_i \quad (k_i\neq 0).$$

Here, c_i is negligible in the limit of $t\to\infty$. However, if one considers $k_i=0$, it just corresponds to constant solution: $F_i=c_i$. Substituting this ansatz to Eq. (13) one can get an equation for k_i as follows:

$$k_i=H_i-\mu\,\mathrm{sign}(k_i). \tag{14}$$

Note that in the $t\to\infty$ limit $\tanh(k_it+c_i)$ depends only on the sign of k_i and can be replaced by $\mathrm{sign}(k_i)$. This equation is solved self-consistently assuming a sign of k_i as follows:

$$k_i=H_i-\mu \quad (k_i>0),$$

$$k_i=H_i+\mu \quad (k_i<0). \tag{15}$$

Since the coefficient of the relaxation term μ is positive, these equations do not have a solution for sufficiently small H_i. In such a case there is only solution $k_i=0$, which leads to time-independent F_i.

Because H_i strongly depends on the frequency of battles, existence of $k_i \neq 0$ varies with the number density of individuals. In general, the absolute value of H_i grows from zero with increasing of the number density ρ. Therefore, at a sufficiently low density, F_i must be constant for any i in a steady state. However, if ρ increases beyond a critical value ρ_c, a new steady state where some of F_i are proportional to t is allowed.

C. Phase transition

In this mean-field analysis we consider that the appearance of a new steady state corresponds to the phase transition in Monte Carlo simulations. In order to correspond the steady states of deterministic equation (6) to phases in Monte Carlo simulations, we introduce two assumptions.

First, we only consider steady states where all F_i with $k_i = 0$ are equivalent. Equation (13) generally has many solutions where $\{F_i\}$ with $k_i = 0$ distributes in a fine balance so that all $dF_i/dt = 0$. However, we assume that these steady states do not appear in Monte Carlo simulations because of strong fluctuations which are not contained in Eq. (6). These fine balances would be easily destroyed by outcome of battles and discreteness of time evolution. Therefore one can observe only averaged values of F_i and X_i.

Second, we introduce a plausible assumption that the phase which appears in Monte Carlo simulation corresponds to *the most hierarchical steady state* among all possible steady states of Eq. (6) at a fixed ρ. Here the most hierarchical steady state is defined as the state where the number of individuals who have the same powers is the least. If power developments are allowed by Eq. (15), power balance is globally unstable against power fluctuations large enough to satisfy $P(\Delta F_{i,j}) \simeq 0$ or $\simeq 1$. Therefore the phases observed in long time simulations must correspond to the most hierarchical phases in the mean-field analysis.

Based on these two assumptions, we can qualitatively explain the phase transitions observed in several Monte Carlo simulations from the mean-field analysis. The most important quantities in this analysis are $\{H_i\}$. Once we know $\{H_i\}$ in the steady state, one can calculate, from Eqs. (7) and (12), the distribution of winning fractions as

$$X_i = \frac{1}{2}\left(\frac{H_i}{\sum_{j\neq i} Q_{i,j}} + 1\right), \tag{16}$$

and also the order parameter σ^2.

As shown later, in order to catch a profile of H_i in the steady state, all one needs to do is to know a profile of H_i in a certain special state. In this special state all individuals have different powers and their powers develop in proportion to t, and we express this special profile as H_i^*. This state can be characterized as the state where proportional coefficients of power k_i are different from each other as follows:

$$k_1 < k_2 < \cdots < k_{N-1} < k_N. \tag{17}$$

Because power differences of any pairs are equal to $\pm\infty$ in the limit of $t \to \infty$, one can reduce, by use of condition (8), Eq. (7) to

$$\frac{dD_i^*}{dt} = \sum_{j=1}^{i-1} Q_{i,j}^*, \quad \frac{dS_i^*}{dt} = \sum_{j=i+1}^{N} Q_{i,j}^*. \tag{18}$$

Here, $Q_{i,j}^*$ expresses $Q_{i,j}$ of the case that there is no individual who has the same power with i or j ($N_i^e = N_j^e = 0$). From this equation, one can calculate H_i^* as $H_i^* = dD_i^*/dt - dS_i^*/dt$.

Based on $\{H_i^*\}$, profiles of H_i for other steady states can be written as follows. Suppose that there are N^e+1 equivalent individuals and the other individuals have different powers such as

$$F_1 < \cdots < F_c = \cdots = F_{c+N^e} < \cdots < F_N. \tag{19}$$

From the first assumption stated at the beginning of this subsection, we only consider the case where power difference $\Delta F_{i,j}$ is equal to $\pm\infty$ or 0.

$$\Delta F_{i,j} = \begin{cases} 0 & (c \le i,j \le c+N^e) \\ \pm\infty & (\text{otherwise}). \end{cases}$$

From this fact and the conservation of battle frequency, Eq. (9), one can calculate H_i as follows:

$$H_i = \begin{cases} \dfrac{1}{N^e+1}\displaystyle\sum_{j=c}^{c+N^e} H_j^* & (c \le i \le c+N^e) \\ H_i^* & (\text{otherwise}). \end{cases} \tag{20}$$

One can easily generalize this relation to any steady states, and the result is quite simple. If there are several equivalent individuals, H_i is an average of H_i^* like Eq. (20), and otherwise H_i is equivalent to H_i^* itself. Therefore $\{H_i^*\}$, which expresses the effect of battles in the most hierarchical steady state, contains the information about H_i of any steady states.

From Eqs. (15) and (20), one can calculate $\{H_i\}$ of the most hierarchical steady state among all possible steady states at any number density as follows. At first, one calculates $\{H_i^*\}$ at a number density of interest. Second, one investigates whether $k_i \neq 0$ can exist by use of Eq. (15). In this investigation one calculates the sign of $H_i^* - \mu$ and $H_i^* + \mu$. If $H_i^* - \mu > 0$ or $H_i^* + \mu < 0$ is satisfied, one assigns a nonzero value to k_i from Eq. (15), and otherwise k_i is equal to zero which corresponds to a solution that F_i is constant. Since H_i^* is a monotonically increasing function of i in general, one typically gets a profile of k_i such as

$$k_1 < \cdots < k_c = \cdots = k_{c+N^e} < \cdots < k_N, \tag{21}$$

where $k_c, k_{c+1}, \ldots,$ and k_{c+N^e} are equal to zero. Finally, one calculates $\{H_i\}$ of this profile by use of Eq. (20). Note that $\{H_i\}$ calculated from this profile is self-consistent to Eq. (15) because of Eq. (20).

This profile is the most hierarchical at this number density. If one considers another profile which is more hierarchical, it does not satisfy Eq. (15). For example, if k_c was less than $k_{c+1} = \cdots = k_{c+N^e}$, H_c is equivalent to H_c^*, and therefore it conflicts with Eq. (15).

Note that we implicitly assumed H_i^* as an increasing function of i in this analysis. In general, it would be possible that H_i^* decreases in a certain region even if we assume inequality (17). However, k_i obtained from such H_i^* becomes a decreasing function of i and it conflicts with the condition (17). In such a case, one must modify H_i^* so that it becomes a nondecreasing function over the entire region and the corresponding profile of k_i is still the most hierarchical among all profiles which lead to nondecreasing H_i. This modification can be done by assuming some individuals have equivalent k_i and by the use of Eq. (20).

The most hierarchical steady state typically changes as a function of the number density as follows. At a sufficient low density, the most hierarchical steady state is $k_i=0$ for all i because H_i^* is so small that finite k_i cannot satisfy Eq. (15). As the number density is increased, absolute values of H_0^* and H_N^* come close to μ, and when either one of the two exceeds μ, the most hierarchical steady state changes. The new steady state contains two sorts of individuals. In one group individuals have $k_i=0$, and in the other group individuals show power growing in which a sign of k_i depends on whether H_0^* or H_N^* exceeds μ in the first. At a sufficient high density, there is another steady state where three sorts of individuals exist: $k_i=0$, $k_i<0$, and $k_i>0$.

Based on the mean-field analysis, we can define three classes of individuals in a steady state: winners, losers, and middle class. Individuals belong to winners when their power is increasing so that $k_i>0$, or belong to losers if $k_i<0$. In the case of $k_i=0$, they belong to the middle class. This definition is different from previous works [11,12], where classification was based on winning fraction and it was simply divided into three equal parts. However, the present definition is better because it can divide individuals depending on qualitative features, the sign of k_i. In this definition, the phase transitions correspond to the appearance or disappearance of classes.

IV. APPLICATION TO MODEL SOCIETIES

A. Random society

In this section we show some applications based on our mean-field analysis. At first we deal with the original model of Bonabeau *et al.* [1]. Hereafter, we describe this original model as the random society in order to emphasize its diffusion rule.

Although this society has already been analyzed by Bonabeau *et al.* based on their mean-field theory [1], there would be some misprints or misinterpretations. Especially, profiles of winning fractions are different from our calculations. In order to clarify our proposition concerning the phase transition at an emergence of hierarchical society, here we repeat the analysis for the random society.

From the definition of the random society, we set $Q_{i,j}$ and $P_{i,j}$ in the mean-field analysis as

$$P_{i,j} = \frac{1}{1+\exp[-\eta(F_i - F_j)]}, \quad Q_{i,j} = 2\frac{\rho}{N-1}. \tag{22}$$

In order to estimate $Q_{i,j}$ we assumed that individuals are distributed uniformly on the lattice space. Based on this assumption the probability that there is a particular individual j on the site where individual i tried to move is equal to $\rho/(N-1)$. Note that there are two situations where a battle between individuals i and j happens: i moves to the site which is already occupied by j and vice versa. Because each individual can move once during a unit time, the battle frequency between i and j can be estimated as Eq. (22). In this random society, $Q_{i,j}$ does not depend on rankings and it is determined only by the number density.

In order to discuss the phase transitions we calculate H_i^*, which is the most important quantity for the transitions as shown in Sec. III. For simplicity we again number individuals in order of strength as Eq. (11) and consider the limit of $N\to\infty$ by introducing a new variable $x=i/N$. Substituting Eq. (22) into Eq. (18), one can readily calculate $H^*(x) = dD^*(x)/dt - dS^*(x)/dt$ as follows:

$$H^*(x) = 4\rho\left(x - \frac{1}{2}\right). \tag{23}$$

Substituting this $H^*(x)$ to H_i of Eq. (15), one can get an equation which finite $k(x)$ must satisfy,

$$k(x) = 4\rho\left(x - \frac{1}{2}\right) - \mu \quad [k(x) > 0],$$

$$k(x) = 4\rho\left(x - \frac{1}{2}\right) + \mu \quad [k(x) < 0]. \tag{24}$$

Since x ranges from 0 to 1, there is no $k(x)$ which satisfies this equation when $\rho < \mu/2$, and the most hierarchical steady state is $k(x)=0$ for the entire region of x. If ρ grows beyond $\mu/2$, $k(x)$ at the edge of x can satisfy Eq. (24) and a new steady state appears. Therefore the critical number density ρ_c of this random society is equal to $\mu/2$.

For $\rho > \rho_c$, the profile of $k(x)$ for the most hierarchical steady state can be written as follows:

$$k(x) = \begin{cases} 2\rho(2x-1)+\mu & (0 \le x \le x_l) \\ 0 & (x_l < x \le x_h) \\ 2\rho(2x-1)-\mu & (x_h < x \le 1). \end{cases} \tag{25}$$

Here the upper and lower bound of the $k(x)=0$ region can be calculated from Eq. (24) as the point where the right-hand side of the equations are equal to zero.

$$x_l = (1-\rho_c/\rho)/2, \tag{26}$$

$$x_h = (1+\rho_c/\rho)/2. \tag{27}$$

Because x_l and x_h are symmetric about $x=0.5$ and $H^*(x)$ is an odd function about $x=0.5$, the average of $H^*(x)$ in the range of $x_l < x \le x_h$ is equal to zero. From this fact one can readily find profiles of $H(x)$ and $X(x)$ by the use of Eqs. (20) and (16), respectively. The results are as follows:

$$H(x) = \begin{cases} 0 & (x_l \le x \le x_h) \\ 2\rho(2x-1) & (\text{otherwise}), \end{cases} \tag{28}$$

and

$$X(x) = \begin{cases} \frac{1}{2} & (x_l \le x \le x_h) \\ x & (\text{otherwise}). \end{cases} \tag{29}$$

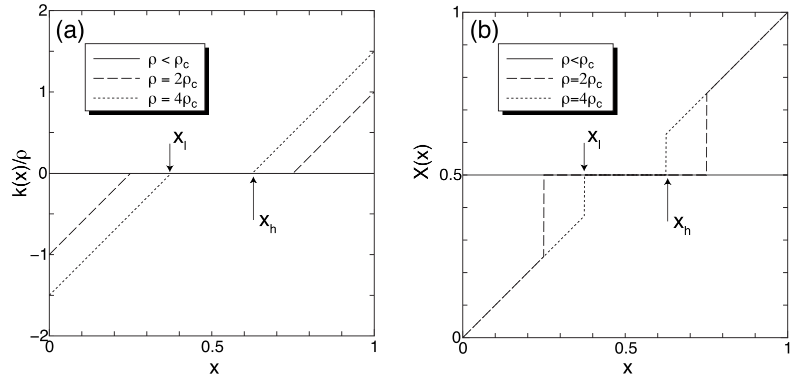

FIG. 1. (a) Coefficients of power evolution $k(x)/\rho$ in the random society as a function of x for a different value of ρ. (b) Winning fraction $X(x)$ in the random society as a function of x for a different value of ρ: $\rho < \rho_c$ (solid line), $\rho = 2\rho_c$ (dashed line), and $\rho = 4\rho_c$ (dotted line). Here ρ_c is equal to $\mu/2$. As a guide for the eyes, we show the positions of x_l and x_h in the case of $\rho = 4\rho_c$ by arrows.

Note that this $X(x)$ is different from the result of Bonabeau *et al.* [1]. They have predicted $X(x) = x + \mu/(4\rho)$ for $0 \leq x < x_l$ and $X(x) = x - \mu/(4\rho)$ for $x_h \leq x \leq 1$ in our notation [16]. As shown later, however, the result of Monte Carlo simulation supports Eq. (29). Therefore we conclude that the result of Bonabeau *et al.* is a misprint or a miscalculation.

Figure 1 shows typical profiles of $k(x)$ and corresponding $X(x)$. While $k(x)$ is equal to zero for an overall range of x for $\rho < \rho_c$, finite $k(x)$ appears and its region grows for $\rho > \rho_c$. In line with the growth of the finite $k(x)$ region, losers and winners appear in winning fraction $X(x)$. Note that while $k(x)$ is a continuous function and does not show overlap between different densities except $k(x) = 0$, $X(x)$ is discontinuous and shows overlap. This is the important prediction of our mean-field analysis.

We conclude this subsection with a comparison of the mean-field analysis and the results of Monte Carlo simulations. Monte Carlo simulation was performed for $N = 4000$ individuals on the square lattice with periodic boundary condition. Variation of number density was controlled by changing lattice size $L \times L$. Each simulation was performed for 6×10^6 Monte Carlo steps. After the first 10^6 Monte Carlo steps, which is sufficiently long in order to reach the steady state, we started recording the number of wins and losses.

Figure 2 shows profiles of winning fraction X_i obtained from Monte Carlo simulation. As predicted by the mean-field analysis, X_i shows the two characteristics: discontinuous jumps and overlaps between different densities. These results support Eq. (29) rather than the result of Bonabeau *et al.*

We plotted the order parameter as a function of the number density in Fig. 3. The mean-field estimation of σ^2 is represented by the use of x_l in Eq. (26) as

$$\sigma^2 = \frac{2}{3}x_l^3 - x_l^2 + \frac{1}{2}x_l. \quad (30)$$

In Fig. 3, the solid curve predicted by the mean-field analysis agrees with the results of Monte Carlo simulation very well. This excellent agreement indicates that the assumption of uniform distribution used to construct $Q_{i,j}$ is valid in Monte Carlo simulation.

B. Timid society

As the second application of the mean-field analysis, we consider the timid society introduced by Odagaki *et al.* [11]. On the basis of Monte Carlo simulation, they have shown that the self-organization of the hierarchical state occurs in two steps as the density is increased. There are three states in this society: one egalitarian and two hierarchical states. A

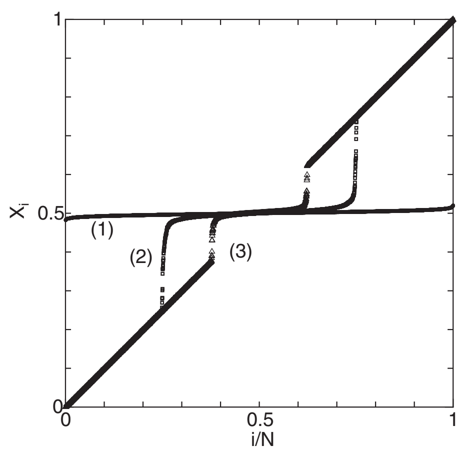

FIG. 2. Winning fraction X_i in the Monte Carlo simulation for 6×10^6 Monte Carlo steps for different densities: (1) $\rho \simeq 0.04$, (2) $\rho = 0.10$, and (3) $\rho \simeq 0.20$. Each simulation was done for $\eta = 5.0$, $\mu = 0.1$, $N = 4000$. For these parameters ρ_c is equal to 0.05.

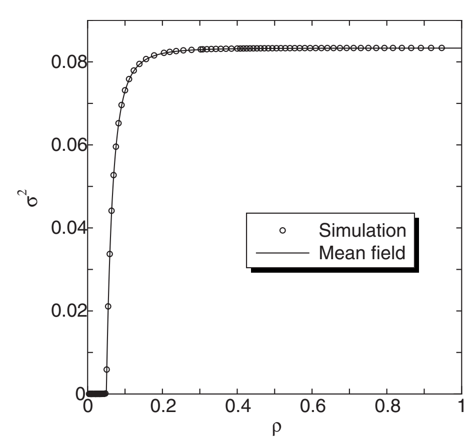

FIG. 3. The order parameter σ^2 as a function of ρ with $\mu = 0.1$. Circles show the result of Monte Carlo simulation with $\eta = 5.0$, $N = 4000$, and $\rho \equiv N/L^2$. A solid curve represents the σ^2 calculated from the mean-field analysis.

difference between the first and the second hierarchical states is whether winners exist or not; all individuals belong to either middle class or losers in the first hierarchical state. Interestingly, while the transition from the egalitarian state to the first hierarchical state is continuous, the transition from the first hierarchical state to the second hierarchical state is discontinuous.

A characteristic of the timid society is the preference of individuals in the diffusion stage. In the timid society, all individuals hope not to fight as much as possible. Therefore it always moves to a vacant site if it exists around them. If all of the nearest neighbor sites are occupied, it moves to a site occupied by an individual whose power is the smallest among the neighbors. When there is more than one individual that has the same and the smallest power, each of them is chosen with equal probability. To be more precise, this society is considered to be timid pacifists. For simplicity, we call this society timid. The other rules for the fighting and relaxation are the same as the random society.

To begin with the mean-field analysis, we consider the form of $Q_{i,j}$. Since $Q_{i,j}$ contains the information of two situations where individual i challenges j to a battle or vice versa, we split $Q_{i,j}$ into two parts:

$$Q_{i,j} = (\tilde{Q}_{i\to j} + \tilde{Q}_{j\to i}), \tag{31}$$

where $\tilde{Q}_{i\to j}$ represents the frequency of battles in which individual i challenges to j. Individual i challenges j when the following three conditions are satisfied. First, all nearest neighbor sites of individual i are occupied, and second, individual j exists on one of these nearest neighbor sites. Finally, individual j must be the weakest among all individuals who occupy the nearest neighbor sites of i. In order to construct $\tilde{Q}_{i\to j}$, here we approximate that every possible configuration of individuals appears with equal probability at any time. Based on this approximation, one can calculate $\tilde{Q}_{i,j}$ as a probability that the specified configurations occur.

$$\tilde{Q}_{i\to j} = 4\frac{\rho^4}{N-1}\frac{{}_{N_j^s}\mathrm{C}_3}{{}_{N-2}\mathrm{C}_3}, \tag{32}$$

where N_j^s represents the number of individuals who are stronger than j, and ${}_n\mathrm{C}_m$ denotes a binomial coefficient. Precisely speaking, N_j^s in the right-hand side of Eq. (32) must not include individual i; however, here we only consider the limit of $N\to\infty$ and neglect this effect. In the same limit the ratio of binomial coefficients can be reduced to a simple form and we get $\tilde{Q}_{i\to j}$ for the timid society as follows:

$$\tilde{Q}_{i\to j} = 4\frac{\rho^4}{N}\left(\frac{N_j^s}{N}\right)^3. \tag{33}$$

Note that the expression (33) can apply only in the case where there are no individuals whose powers are the same as j. If there are some individuals whose powers are equivalent to each other, $\tilde{Q}_{i\to j}$ is calculated by use of Eq. (9) as follows:

$$\tilde{Q}_{i\to j} = \frac{\rho^4}{N}\left(\frac{N}{N_j^e}\right)\left[\left(\frac{N_j^s+N_j^e}{N}\right)^4 - \left(\frac{N_j^s}{N}\right)^4\right], \tag{34}$$

where N_j^e represents the number of individuals whose powers are the same as individual j, and we neglect unity compared with N_j^s and N_j^e. Of course one can also lead this expression from a calculation of the probability such as the derivation of Eq. (32).

Next we calculate H_i^* in order to investigate the transitions. We again employ ordering (11), and for simplicity we consider the limit of $N\to\infty$ by use of a new variable $x \equiv i/N$. Note that $N_i^s/N=(1-x)$. Substituting $Q_{i,j}$ into Eq. (18), one can get $H^*(x)$ as follows:

$$\begin{aligned} H^*(x) &= 4\rho^4\left\{\int_0^x dx'[(1-x')^3+(1-x)^3] \right.\\ &\quad \left. - \int_x^1 dx'[(1-x')^3+(1-x)^3]\right\} \\ &= \rho^4[(10x-6)(1-x)^3+1]. \end{aligned} \tag{35}$$

Note that this $H^*(x)$ is a decreasing function in the range of $7/10 < x \leq 1$. As mentioned in Sec. III, this $H^*(x)$ is inappropriate to determine the most hierarchical steady state. Therefore we modify $H^*(x)$ so that it becomes a nondecreasing function in the entire region of $0\leq x\leq 1$ by assuming that beyond a certain x_c, all individuals are equivalent. The new $H^*(x)$, which we denote as $\tilde{H}^*(x)$, is calculated from Eq. (20), and therefore conditions for x_c in order for $\tilde{H}^*(x)$ to become a nondecreasing function are as follows:

$$\frac{dH^*(x)}{dx} \geq 0 \quad (0\leq x\leq x_c),$$

$$\begin{aligned} H^*(x_c) &\leq \frac{1}{1-x_c}\int_{x_c}^1 H^*(x')dx' \\ &= \rho^4[(2x_c-1)(1-x_c)^3+1]. \end{aligned} \tag{36}$$

Among x_c, which satisfy the above conditions, we choose the largest one so that obtained $\tilde{H}^*(x)$ can be the most hierarchical. We find $x_c=5/8$ and therefore get $\tilde{H}^*(x)$ as follows:

$$\tilde{H}^*(x) = \begin{cases} H^*(x) & \left(0\leq x\leq \frac{5}{8}\right) \\ \tilde{H}_c^* & \left(\frac{5}{8} < x \leq 1\right), \end{cases} \tag{37}$$

where $\tilde{H}_c^*$ is a constant,

$$\tilde{H}_c^* = \rho^4[(2x_c-1)(1-x_c)^3+1] = \rho^4\left[\frac{1}{4}\left(\frac{3}{8}\right)^3+1\right]. \tag{38}$$

As shown later, this modification plays an essential role in the discontinuous transition.

Figure 4 shows $\tilde{H}^*(x)$ as a function of x. Note that $\tilde{H}^*(x)$ is proportional to ρ^4 and therefore $\tilde{H}^*(x)/\rho^4$ does not depend on ρ. This $\tilde{H}^*(x)$ has two characteristics which are essential

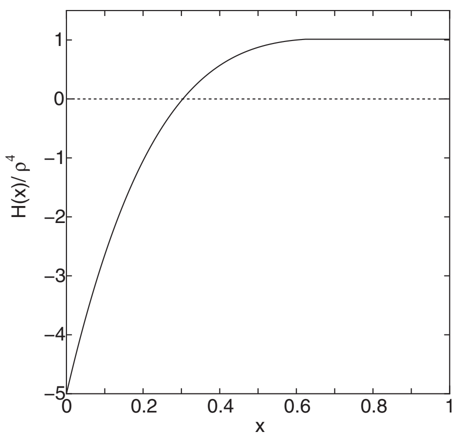

FIG. 4. $\tilde{H}^*(x)/\rho^4$ of the timid society as a function of x (solid curve). The dotted line shows $\tilde{H}^*(x)=0$ as a guide. The minimal value of $\tilde{H}^*(x)/\rho^4$ is equal to -5 at $x=0$ and the maximal value is $(2x_c-1)(1-x_c)^3+1$ in $x \geq x_c$, where x_c is equal to $5/8$.

to understand the transitions. First, the absolute value of the minimum is larger than the maximum, and second, $\tilde{H}^*(x)$ shows the maximal value not only at a certain point but also in the finite range, $5/8 \leq x \leq 1$. These two aspects are different from the random society where absolute values of minimum and maximum are the same and they appear only at a certain point, $x=0$ and $x=1$, respectively.

Substituting $\tilde{H}^*(x)$ into Eq. (15) and comparing both sides of the equations, one can determine the most hierarchical steady state at any ρ. At a sufficiently low density, the most hierarchical steady state is $k(x)=0$ for $0 \leq x \leq 1$. As ρ is increased, a solution which satisfies the equation for $k(x)<0$ appears. The critical density ρ_{c_1} for this transition is the density at which the minimal value of $\tilde{H}^*(x)$ is equal to $-\mu$.

$$\rho_{c_1} = \left(\frac{\mu}{5}\right)^{1/4}. \tag{39}$$

Because of the difference between the minimal and the maximal values of $\tilde{H}^*(x)$, a profile of $k(x)$ is nonpositive for the entire region.

$$k(x) = \begin{cases} \tilde{H}^*(x) + \mu & (0 \leq x \leq x_l) \\ 0 & (x_l < x \leq 1), \end{cases} \tag{40}$$

where x_l is a solution of equation $\tilde{H}^*(x_l)+\mu=0$ at a density interested. When ρ grows much further, a solution which satisfies the equation for $k(x)>0$ appears. At this transition, critical density ρ_{c_2} is defined as the density where the maximal value of $\tilde{H}^*(x)$ is the same as μ. From Eq. (38), we find ρ_{c_2} as

$$\rho_{c_2} = \left[\frac{\mu}{(2x_c-1)(1-x_c)^3+1}\right]^{1/4}. \tag{41}$$

At a density beyond ρ_{c_2}, the profile of $k(x)$ can be written as follows:

$$k(x) = \begin{cases} \tilde{H}^*(x) + \mu & (0 \leq x \leq x_l) \\ 0 & (x_l < x \leq x_h) \\ \tilde{H}^*(x) - \mu & (x_h < x \leq 1), \end{cases} \tag{42}$$

where x_l and x_h are solutions of equations $\tilde{H}^*(x_l)+\mu=0$ and $\tilde{H}^*(x_h)-\mu=0$, respectively. Note that while only "one" individual $x=0$ has negative $k(x)$ at the first transition, at the second transition a lot of individuals who belong to $x_c < x \leq 1$ have positive $k(x)$. Therefore one expects that the corresponding order parameter changes discontinuously at the second transition. This expectation will be checked after we calculate $X(x)$ and the order parameter.

If we increase ρ beyond ρ_{c_2}, both x_l and x_h come close to the zero of $\tilde{H}^*(x)$, which lies between x_l and x_h, and in the limit of $\rho \to \infty$ they merge. Although ρ should be limited by one in real systems, we can also interpret ρ as a parameter which measures the effect of battles. From this point of view, the system in the limit of $\rho \to \infty$ corresponds to the system with no relaxation because the effect of relaxation is negligible compared to battle effects.

Once one knows a profile of $k(x)$ for the most hierarchical steady state, one can calculate $H(x)$ and $X(x)$ by Eqs. (20) and (16). Note that in the calculation of $X(x)$, one must use Eq. (34) instead of Eq. (33) for x corresponding to flat $k(x)$. The results are written in the Appendix.

Figure 5 shows typical profiles of $k(x)$ and $X(x)$. In the

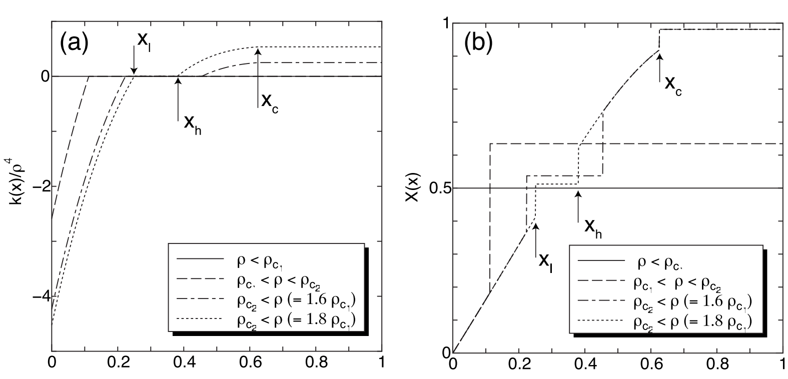

FIG. 5. (a) Coefficients of power evolution $k(x)/\rho^4$ in the timid society as a function of x for different values of ρ. (b) Winning fraction $X(x)$ in the timid society as a function of x for different values of ρ: $\rho < \rho_{c_1}$ (solid curve), $\rho = 1.2\rho_{c_1}$ (dashed curve), $\rho = 1.6\rho_{c_1} > \rho_{c_2}$ (dot-dashed curve), and $\rho = 1.8\rho_{c_1} > \rho_{c_2}$ (dotted curve). Here ρ_{c_1} and ρ_{c_2} are represented in Eqs. (39) and (41), respectively. As a guide for the eyes, we show the positions of x_l, x_h, and x_c in the case of $\rho = 1.8\rho_{c_1}$ by arrows.

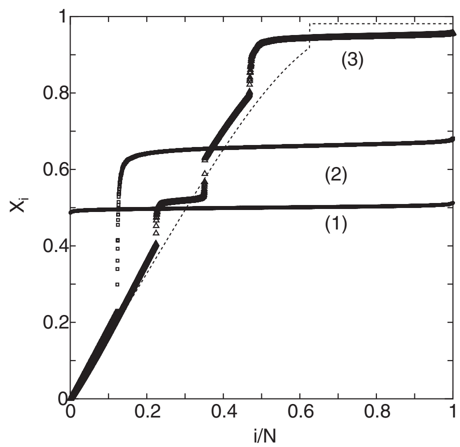

FIG. 6. Winning fraction X_i in the Monte Carlo simulation of the timid society for 6×10^6 Monte Carlo steps for different densities: (1) $\rho\simeq0.30$, (2) $\rho\simeq0.51$, and (3) $\rho\simeq0.67$. The dotted curve shows $X(x)$ of the mean-field analysis for $\rho\rightarrow\infty$. Each simulation was done for $\eta=5.0$, $\mu=0.1$, and $N=4000$. At these parameters ρ_{c_1} and ρ_{c_2} predicted by mean-field analysis are approximately equal to 0.37 and 0.56, respectively.

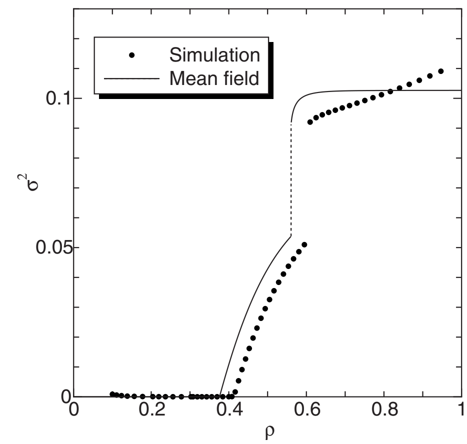

FIG. 7. The order parameter σ^2 of the timid society as a function of ρ with $\mu=0.1$. The circles show the results of Monte Carlo simulation with $\eta=5.0$, $N=4000$, and $\rho\equiv N/L^2$. The solid curve represents the σ^2 calculated from the mean-field analysis. The dotted line represents the discontinuous jump of σ^2 at $\rho=\rho_{c_2}$ in the mean-field analysis.

first hierarchical steady state ($\rho_{c_1}\leq\rho<\rho_{c_2}$), $k(x)$ is negative or zero, while $k(x)$ can be positive in the second hierarchical steady state ($\rho_{c_2}\leq\rho$). This fact just corresponds to the characteristic shown in the Monte Carlo simulations where all individuals belong to either middle class or losers in the first hierarchical society, and winners appear in the second hierarchical society. In response to the change of $k(x)$, the profile of $X(x)$ changes qualitatively. As in case of the random society, $X(x)$ changes discontinuously at boundaries of flat regions, $x=x_l,x_h,x_c$, and $X(x)$ for different densities overlap in the region corresponding to $k(x)\neq0$. However, unlike with the random society, $X(x)$ corresponding to $k(x)=0$ is not equal to $1/2$ and its value depends on a number density. Note that $X(x)$ for $x_c<x\leq1$ jumps close to 1 at the transition in ρ_{c_2}. This simultaneous change causes discontinuous change in the order parameter. As mentioned before, if we increase ρ beyond ρ_{c_2} the distance between x_h and x_l becomes shorter, and in the limit of $\rho\rightarrow\infty$ middle class, the region of $k(x)=0$ disappears. In this limit, the discontinuous jumps of $X(x)$ at x_l and x_h no longer exist.

In Fig. 6, we plotted profiles of winning fraction $X(x)$ obtained from Monte Carlo simulation. Simulations were performed with the same parameters as the random society: $N=4000$ in the square lattice with periodic boundary condition and each simulation was performed for 6×10^6 Monte Carlo (MC) steps. The shapes of X_i profiles qualitatively agree with the results of the mean-field analysis. However, the value of x_c, which is approximately 0.5, is largely different from the mean-field prediction, $x_c=0.625$. This discrepancy would be caused by the approximation used to estimate $Q_{i,j}$, in which every possible configuration of individuals appears with equal probability at any time. Because of the timid moving rule, occurrences of each configuration are not equivalent in Monte Carlo simulations.

Figure 7 shows the order parameter as a function of number density. The solid curve predicted by the mean-field analysis qualitatively explains the first continuous transition and the second discontinuous one as expected from the profile of $X(x)$. However, there are several quantitative discrepancies between the Monte Carlo simulation and the mean-field prediction. At first, the mean-field analysis underestimates two critical densities, ρ_{c_1} and ρ_{c_2}. Furthermore, reflecting the discrepancy of $X(x)$, values of σ^2 in the second hierarchical phase are different between the simulation and the mean-field analysis.

C. Brave society

As the last example, we discuss a brave pacifist or simply *brave* society. In this society, every individual favors avoiding fighting if possible, as in the timid society. However, when they cannot avoid fighting they try to fight with the strongest one among individuals who are on the nearest neighbor sites. So far, this model has not been investigated by Monte Carlo simulation. But, applying the mean-field analysis, we can predict that it shows two-step phase transitions in the same way as the timid society.

The mean-field analysis of this society can be made in the same way as the timid society. Based on the same assumption with the timid society, $\tilde{Q}_{i\rightarrow j}$ is defined as

$$\tilde{Q}_{i\rightarrow j}=4\frac{\rho^4}{N}\left(\frac{N_j^w}{N}\right)^3, \tag{43}$$

where N_j^w represents the number of individuals who are weaker than j and we considered the limit of $N\rightarrow\infty$. Note that the only difference between the timid and the brave societies is that either $\tilde{Q}_{i\rightarrow j}$ depends on N_j^s or N_j^w.

The corresponding H_i^* can be calculated by substituting Eq. (43) into Eq. (18). Here we use the numbering as in Eq. (11) and the new variable $x=i/N$ in the $N\rightarrow\infty$. In this notation $N_i^w/N=x$ and the result is given by

$$H^*(x)=\rho^4[x^3(10x-4)-1]. \tag{44}$$

As in the case of timid society, this $H^*(x)$ is not a monotonically increasing function. Note, however, that a decrease of $H^*(x)$ appears in the vicinity of $x=0$. It is in contrast to the timid society where the decrease appears in the vicinity of $x=1$. In order to discuss phase transitions, we modify $H^*(x)$ so as to become a nondecreasing function by assuming individuals are equivalent in $0 \leq x \leq x_c$. Again we describe this modified $H^*(x)$ as $\tilde{H}^*(x)$. The condition x_c must satisfy is

$$\frac{dH^*(x)}{dx} \geq 0 \quad (x_c \leq x \leq 1),$$

$$H^*(x_c) \geq \frac{1}{x_c}\int_0^{x_c} H^*(x')dx' = \rho^4[x_c^3(2x_c-1)-1]. \tag{45}$$

We choose the least x_c so that modified $H^*(x)$ becomes the most hierarchical, and the least one is $x_c=3/8$. Therefore one can calculate $\tilde{H}^*(x)$ as follows:

$$\tilde{H}^*(x) = \begin{cases} \tilde{H}_c^* & \left(0 \leq x < \frac{3}{8}\right) \\ H^*(x) & \left(\frac{3}{8} \leq x \leq 1\right). \end{cases} \tag{46}$$

Here $\tilde{H}_c^*$ is a constant such as

$$\tilde{H}_c^* = \rho^4[x_c^3(2x_c-1)-1] = -\rho^4\left[\frac{1}{4}\left(\frac{3}{8}\right)^3 + 1\right].$$

Interestingly, $\tilde{H}_c^*$ is just a minus sign of Eq. (38). One can find this symmetry in $H^*(x)$ of Eq. (44) by replacing x with $(1-x)$. Therefore one would expect that the transition phenomenon in the brave society is almost the same as the timid society.

Actually, transitions happen by two steps and the second one is a discontinuous transition in this brave society. In Fig. 8, we plotted $\tilde{H}^*(x)$. From the shape of $\tilde{H}^*(x)$ one can determine the most hierarchical state at any density as follows. At sufficient low density, only the state $k(x)=0$ for the entire range of x is allowed, and therefore all individuals are equivalent. When ρ is increased beyond $\rho_{c_1}=(\mu/5)^{1/4}$, the first hierarchical state appears. In this state, every individual belongs to $k(x)=0$ or $k(x)>0$; namely, some winners appear. Note that this state is in contrast to the first hierarchical state of the timid society, where every individual belongs to $k(x)=0$ or $k(x)<0$ so that there is no winner. When ρ is increased further and it exceeds ρ_{c_2} defined in Eq. (41), the second hierarchical state is allowed. In the second hierarchical state $k(x)<0$ emerges, and at the transition all individuals who belong to $0 \leq x < x_c$ change to losers. Therefore the transition from the first hierarchical state to the second one shows a discontinuous jump.

Because of the symmetry between the brave society and the timid society, the order parameter σ^2 behaves exactly the same as the timid society. Therefore, if one observes only σ^2, these two societies cannot be distinguished. The differences appear in profiles of $k(x)$ and $X(x)$.

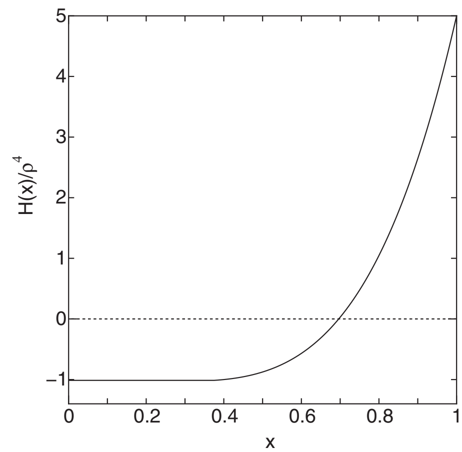

FIG. 8. $\tilde{H}^*(x)/\rho^4$ of the brave society as a function of x (solid line). The dotted curve shows $\tilde{H}^*(x)=0$ as a guide. The maximal value of $\tilde{H}^*(x)/\rho^4$ is equal to 5 at $x=1$ and the minimal value is $-(2x_c-1)x_c^3-1$ in $x \leq x_c$, where x_c is equal to 3/8.

In Fig. 9, we show profiles of X_i obtained from Monte Carlo simulations. Simulations were performed with the same parameters as the random society and the timid society. As expected from the mean-field analysis, if we rotate profiles around $(i/N, X_i)=(0.5, 0.5)$, qualitative features of profiles are similar to those of the timid society (please see Fig. 6). However, we also find that the profile of $\rho \simeq 0.67$ is quantitatively different from that of the timid society. Especially, the fraction of the lowest classes is less than the mean-field prediction while in the timid society the counterpart, which is the fraction of the highest class, is more than the mean-field prediction. The origin of this discrepancy would be a difference of the microdynamics between the winner and the loser of each fight: the winner can keep its position while the loser loses its position. This difference was not contained in the mean-field analysis.

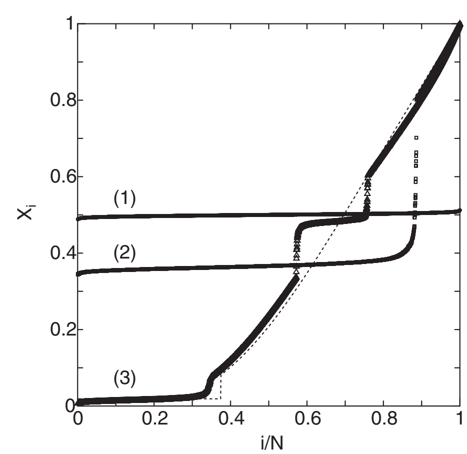

FIG. 9. Winning fraction X_i in the Monte Carlo simulation of the brave society for 6×10^6 Monte Carlo steps for different densities: (1) $\rho \simeq 0.30$, (2) $\rho \simeq 0.51$, and (3) $\rho \simeq 0.67$. The dotted curve shows $X(x)$ of the mean-field analysis for $\rho \to \infty$. Each simulation was done for $\eta=5.0$, $\mu=0.1$, and $N=4000$.

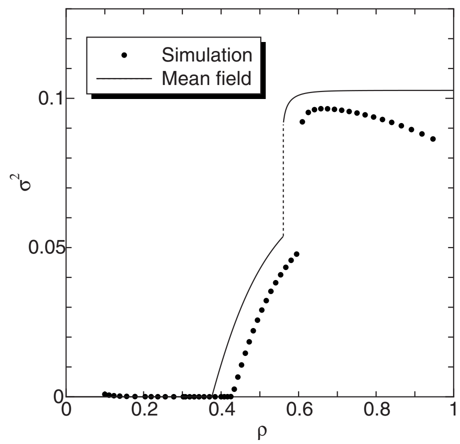

FIG. 10. The order parameter σ^2 of the brave society as a function of ρ with $\mu=0.1$. The circles show the results of Monte Carlo simulation with $\eta=5.0$, $N=4000$, and $\rho\equiv N/L^2$. The solid curve represents the σ^2 calculated from the mean-field analysis. The dotted line represents the discontinuous jump of σ^2 at $\rho=\rho_{c_2}$ in the mean-field analysis.

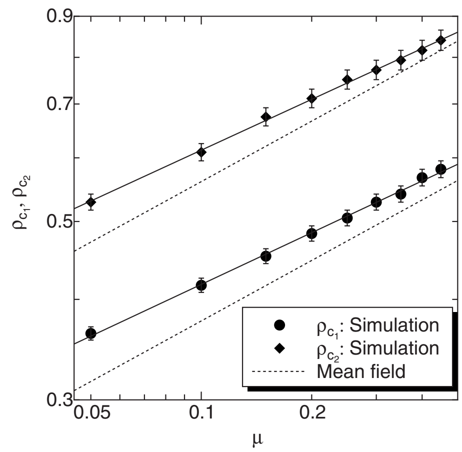

FIG. 11. Critical number densities ρ_{c_1} and ρ_{c_2} as functions of μ on log-log scale. The circles and the diamonds show ρ_{c_1} and ρ_{c_2} obtained from the Monte Carlo simulation with $\eta=5.0$ and $N=4000$, respectively. The dotted lines represent ρ_{c_1} and ρ_{c_2} calculated from the mean-field analysis. The solid lines represent the power-law fittings of the simulation results.

Figure 10 shows the order parameter of the brave society as a function of the number density. The same as the timid society, transitions occur in the two steps in the Monte Carlo simulation of the brave society, and the first transition is continuous while the second one is discontinuous. This similarity agrees with the mean-field analysis which predicts the same order parameter for the timid society and the brave society as a function of the number density. Quantitative differences between the timid society and the brave society come from the symmetry breaking of microdynamics, which was mentioned in the last paragraph.

V. DISCUSSION AND SUMMARY

In this paper, we have presented a mean-field analysis for the phase transitions which appear in the stochastic models of hierarchical societies. Our mean-field analysis can be applied to generalized Bonabeau models where a trend of society is introduced as a complex diffusion rule. In the mean-field analysis, one can explain the phase transition by $H^*(x)$, which represents a gain from battles at a given density. The phase transition happens at a density where effects of relaxation and fighting balance, and a hierarchical phase is characterized as the state where the power of some individuals grows in proportion to the time.

We also applied this mean-field analysis into the timid society introduced by Odagaki and Tsujiguchi [11]. The mean-field analysis nicely explains the two-step phase transitions and profiles of the winning fraction in each phase, at least qualitatively. In the first hierarchical phase all individuals belong to losers or middle class, and in the second hierarchical phase winner individuals appear. The mean-field analysis predicts that the transition to the first hierarchical phase is continuous, while the transition to the second hierarchical phase is rigorously discontinuous.

The discontinuous transition was caused by the fact that $H^*(x)$ decreases for large x. In the physical meaning, this fact represents that if an individual is too strong, his/her gain becomes lower than weaker individuals. This effect comes from the characteristics of timid society; individual favors to fight weaker ones as possible. For stronger individuals, this trend decreases the chance to fight. Although the winning probability of stronger individuals is larger than that of weaker individuals, the number of fights of stronger individuals is less than that of weaker individuals. As the competition of these opposite effects, a gain $H^*(x)$ decreases for individuals who are too strong, and he/she cannot keep his/her ranking anymore.

By use of the mean-field analysis, we also calculated the transition densities as functions of μ. From Eqs. (39) and (41), transition densities show power-law dependence on μ. In Fig. 11, we plotted ρ_{c_1} and ρ_{c_2} as a function of μ. The mean-field analysis quantitatively underestimates both of them, but qualitatively it agrees with a power-law increase of the simulation results. One can nicely fit the data points by a power-law function, and we found that exponents equal to 0.21 for ρ_{c_1} and ρ_{c_2}. These values are less than the mean-field exponent 0.25 as shown in Eqs. (39) and (41). The main reason for this discrepancy is the fact that individuals favor moving to empty sites in order to avoid fighting and therefore a probability that an individual is surrounded by all four sides is lower than ρ^4 in the simulation.

Note that our mean-field analysis predicts that critical densities do not depend on η that appears in the winning probability (1). For all of $P(\Delta F_{i,j})$ that satisfies the condition (8), the mean-field analysis leads to the same results about steady states. This does not agree with several results of Monte Carlo simulations where critical densities increase as η decreases [1,11]. However, this discrepancy would be caused from the fact that the simulation time is not long enough to reach a true steady state. In order to reach a hierarchical steady state, power difference must develop large enough so that $P_{i,j}$ can be assumed to be zero or unity. The criterion for this condition is $\eta \Delta F \gg 1$ and therefore it needs a longer simulation time for a smaller value of η. Actually, we found that the critical density of the timid society for $\eta = 0.05$ approaches that of $\eta = 5.0$ as the time of Monte Carlo simulation becomes much longer.

The analysis for the timid society can be easily generalized to higher dimensions. Because of the mean-field approximation, even in higher dimensions, the phase transition happens qualitatively in the same manner. The critical densities in general d-dimensional cubic lattices are as follows:

$$\rho_{c_1}(d) = \left[\frac{\mu}{1+2d}\right]^{1/2d},$$

$$\rho_{c_2}(d) = \left[\frac{\mu}{(1-x_c)^{2d-1}(2x_c-1)+1}\right]^{1/2d},$$

where $x_c = (1+2d)/(4d)$. So far, no simulation has been made in higher dimensions. Investigation of the timid society in higher dimensions and complex networks is an open problem.

In this paper, the mean-field analysis was based on equations of individual powers. Instead of this approach, one can consider the time evolution of distribution function, such as the work of Ben-Naim *et al.* [14,15]. This version of mean-field analysis would be appropriate for stability analysis. The development of such studies will deepen understanding of the emergence of hierarchy from the aspect of a phase transition.

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APPENDIX

Here we calculate $H(x)$ and $X(x)$ of the timid society in the hierarchical state. For $\rho_{c_1} < \rho < \rho_{c_2}$, the profile of $k(x)$ is represented as Eq. (40). Because $k(x)$ is flat in $x_l < x \leq 1$, corresponding $H(x)$ is given by the average of $\tilde{H}^*(x)$ in this region such as Eq. (20). Therefore, we obtain $H(x)$ as

$$H(x) = \begin{cases} \tilde{H}^*(x) & (0 \leq x \leq x_l) \\ \rho^4[(2x_l-1)(1-x_l)^3+1] & (x_l < x \leq 1). \end{cases} \tag{A1}$$

One can calculate $X(x)$ by use of Eq. (16) in the limit of $N \to \infty$. Note that if $k(x)$ has a flat region, one must use Eq. (34) instead of Eq. (33) as $\tilde{Q}_{i\to j}$ for j/N in the flat region. The summation of $Q_{i,j}$ in Eq. (16) is easily calculated in the limit of $N \to \infty$. If $x = i/N$ does not belong to the flat region of $k(x)$, it is given as

$$\sum_{j\neq i} Q_{i,j} = \sum_{j\neq i} (\tilde{Q}_{j\to i} + \tilde{Q}_{i\to j}) \to 4\rho^4 \int_0^1 [(1-x)^3 + (1-x')^3]dx' = \rho^4[4(1-x)^3+1]. \tag{A2}$$

Note that $\Sigma_{j\neq i}\tilde{Q}_{i\to j}$ is always equal to ρ^4 regardless of whether N_i^e is equal to zero or not. This summation simply represents the frequency of battles caused by i. The difference of the total battle frequency between individuals comes from the term $\Sigma_{j\neq i}\tilde{Q}_{j\to i}$, which represents the case that someone challenges i. In the timid society, the stronger individual is hardly challenged by the other individual. Therefore the total battle frequency is a decreasing function of i. If $k(x)$ is constant in the region $x_1 < x < x_2$, the summation of $Q_{i,j}$ for x in this region is represented as

$$\sum_{j\neq i} Q_{i,j} \to \rho^4\left[\frac{(1-x_1)^4-(1-x_2)^4}{x_2-x_1}+1\right]. \tag{A3}$$

Here, the difference between Eqs. (A2) and (A3) is caused only by the difference of $\tilde{Q}_{j\to i}$. Substituting Eqs. (A1)–(A3) into Eq. (16), one can get $X(x)$ as

$$X(x) = \begin{cases} \frac{1}{2}\left[\frac{(10x-6)(1-x)^3+1}{1+4(1-x)^3}+1\right] & (0 \leq x \leq x_l) \\ \frac{1}{2}\left[\frac{(2x_l-1)(1-x_l)^3+1}{1+(1-x_l)^3}+1\right] & (x_l < x \leq 1). \end{cases} \tag{A4}$$

At a density higher than ρ_{c_2}, $k(x)$ is constant in $x_l < x \leq x_h$ and $x_c < x \leq 1$, and therefore $H(x)$ and $X(x)$ are calculated as

$$H(x) = \begin{cases} \rho^4\left[\frac{(2x_l-1)(1-x_l)^4-(2x_h-1)(1-x_h)^4}{x_h-x_l}+1\right] & (x_l < x \leq x_h) \\ \tilde{H}^*(x) & (\text{otherwise}), \end{cases} \tag{A5}$$

and

$$X(x)=\begin{cases}\frac{1}{2}\left[\frac{(2x_l-1)(1-x_l)^4-(2x_h-1)(1-x_h)^4+(x_h-x_l)}{(x_h-x_l)+(1-x_l)^4+(1-x_h)^4}+1\right] & (x_l<x\le x_h)\\ \frac{1}{2}\left[\frac{(2x_c-1)(1-x_c)^3+1}{1+(1-x_c)^3}+1\right] & (x_c<x\le 1)\\ \frac{1}{2}\left[\frac{(10x-6)(1-x)^3+1}{1+4(1-x)^3}+1\right] & (\text{otherwise}).\end{cases} \tag{A6}$$

The order parameter can be calculated from Eqs. (A4) and (A6) as

$$\sigma^2=\int_0^1 (X(x)-\langle X\rangle)^2dx, \tag{A7}$$

where $\langle X\rangle$ is defined as

$$\int_0^1 X(x)dx. \tag{A8}$$

Because $X(x)$ is a rational function, one can analytically calculate the integral on the right-hand side of Eq. (A7) and σ^2 is represented as a function of x_l, x_h, and $x_c=5/8$. Note that x_l and x_h are the solutions of equations $\tilde{H}^*(x_l)+\mu=0$ and $\tilde{H}^*(x_h)-\mu=0$. Since $\tilde{H}^*(x)$ is a quartic function, one can solve these equations analytically and get x_l and x_h as a function of ρ. Therefore one can describe the order parameter as a function of ρ.

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